A Performance Comparison of Adaptive Operator-Customized Wavelet Basis and Adaptive H-refinement Methods for 2-D Finite Element Analysis

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This paper compares the performance of the popular adaptive H-refinement (HR) technique for the Finite Element Method (FEM) with the Operator-Customized Wavelet Basis (OCWB) FEM in its adaptive version. The latter is somehow an evolution of the HR method with the use of second generation wavelet theory, which allows the solution to be decoupled between iterations, decreasing significantly the total number of degrees of freedom and consequently, the processing time. Conversely, this procedure increases the algorithm complexity, that being the reason why there are such few applications on the subject. Like the HR method, adaptive OCWB can be programmed with various strategies. Since the results are shown in terms of processing time on a regular PC, both algorithms have been developed with similar structures.

***Index Terms*—Adaptive algorithms, adaptive mesh refinement, finite element analysis, wavelet transforms.**

I. Introduction

H-REFINEMENT (HR) is a popular technique created in the 1980s and still used on state of the art Finite Element Method (FEM) simulation softwares [1]. It's an adaptive method with qualities such as flexibility, which allow various situations to be simulated without significant increase in formulation complexity. However, the fact that today's softwares still rely on such longstanding method makes research on FEM optimization popular.

Wavelet theory has been applied to FEM in several situations but only with its second generation [2] results became truly significant [3] - it made possible the creation of functions fully customized to different needs. Named Operator-Customized Wavelet Basis (OCWB) [4], in this FEM case, the wavelets are custom designed to decouple the stiffness matrix between scales, or iterations, factor that decreases significantly the computational cost of the algorithm. Another interesting characteristic of this method is that the solution is the local error itself, which is desirable for adaptive methods. Although the properties mentioned here are interesting, the functions are completely dependant on the problems operator and geometry, factor that increases significantly the complexity of the algorithm for more general cases [5].

The drawbacks linked with such interesting improvement motivates comparison, so the feasibility of theory development can be accessed.

Both algorithms are applied to a 2-D Poisson Equation problem in order to validade the work. Since the results are given in processing time, it is important to note that the adaptive algorithms compared here have similar structures.

II. Algorithms' Structures

The OCWB FEM has as main feature the transformation of a hierarchical system of equations – which comes from FEM adaptivity – described as

$$\begin{bmatrix} G_{IN} & C_{IN,IN+1} & \cdots & C_{IN,FI} \\ C_{IN+1,IN} & G_{IN+1} & & C_{IN+1,FI} \\ \vdots & & \ddots & \\ C_{FI,IN} & C_{FI,IN+1} & & G_{FI} \end{bmatrix} \begin{bmatrix} \alpha_{IN} \\ \alpha_{IN+1} \\ \vdots \\ \alpha_{FI} \end{bmatrix} = \begin{bmatrix} e_{IN} \\ e_{IN+1} \\ \vdots \\ e_{FI} \end{bmatrix}, \quad (1)$$

where sub-matrices G and C contain interactions of functions representing the same and different iterations, respectively. Vectors e represent the problem's excitation and α the unknowns. Subscripts IN and FI represent initial and final iterations, respectively. The aforementioned transformation comprises vanishing sub-matrices C. In other words, the system becomes decoupled between scales, meaning that in every iteration only the degrees of freedom added on the previous step will compose the system.

About the adaptive structure, both algorithms are based on element refinement. That is, when critical, an element is split into 4 by creating and connecting nodes on all three sides – note that triangular meshes are used. The difference between both lies on the decision whether or not to refine an element.

For the OCWB case, the solution on a given node represents the error, or detail, on it. So, as proposed in [6], a practical scheme would be to use this detail for element refinement decision. Although it seems a wise choice, tests proven there is a more interesting approach: to use, instead, the wavelet coeficients, which is the direct result of the OCWB FEM system (see [5]). In that case, the difference between the adaptive and non-adaptive solutions is always below the detail threshold. For a few reasons, that doesn't happen on the previous procedure. This is an important characteristic because it enables the solution to have the desired accuracy. In other words, the error resulting from the adaptive algorithm is always below the chosen detail threshold. For that reason, the latter is used in this paper.

For the HR case, a simple and popular scheme is used: if the integral of the solution's gradient on the element is bigger than a detail threshold, the element is refined for the next iteration.

An important observation is that, since using the same detail threshold for both methods won't result on the same mesh refinement, different thresholds were used so the number of degrees of freedom added in each iteration were almost identical, thus assuring a fairer comparison. The equivalent thresholds were obtained after several tests.

III. RESULTS

The problem simulated in this paper is the simple 2-D Poisson equation on square homogeneous medium. As source, several impulses (Green's functions) were distributed along the domain. Fig. 1 shows the solution obtained using adaptive OCWB with detail threshold equal to 0.03 and 9 iterations. Fig. 2 shows the resulting meshes in this case and also with HR using a detail threshold of 0.00058, which produces a result similar in precision. To confirm accuracy similarities, the numbers of degrees of freedom added after each iteration for various equivalent detail thresholds are shown on Table 1. The processing time of both algorithms for those thresholds and for different numbers of iterations are shown on Table 2. These values were obtained using MATLAB's sparse matrix operations. It is important to note that full memory is preallocated for speed gain when assembling a matrix, fact which explains lack of memory in some cases. It is later set to sparse representation.

Although the complexity of the OCWB algorithm for more general cases increases significantly, the computational gain is clear. As shown in Table 2, the processing time of the adaptive OCWB is in one situation approximately 30 times faster than the HR method, and it is expected to be even faster for cases where it is necessary further refinement.

As already mentioned, OCWB enables the system to be decoupled between scales. In other words, in every iteration, only the added degrees of freedom compose the system to be solved. An example: as shown in Table 1, with DT=0.03 (0.00058), after the fifth iteration, 2403 degrees of freedom are added. The dimension of the matrix to be inverted will be 2403x2403 on the next step. For the HR case, the system to be solved by this iteration will have 16882 – which is the sum of all iterations thus far - degrees of freedom.

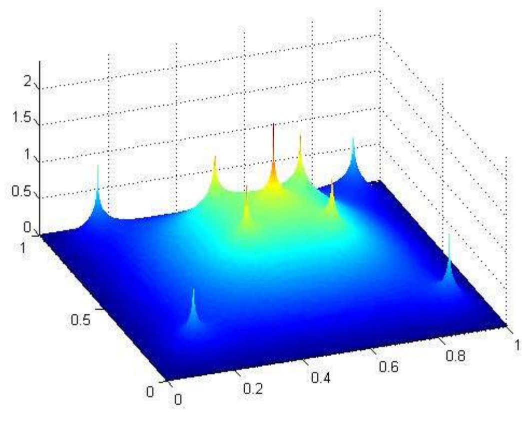
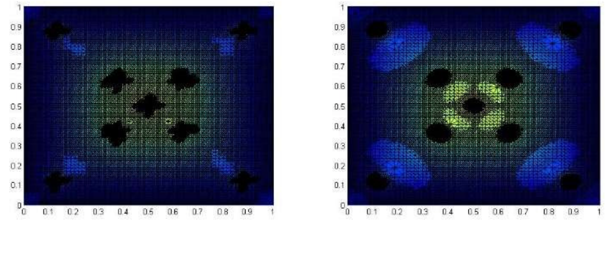


Fig. 1. Solution of the proposed problem obtained using adaptive OCWB FEM with detail threshold of 0.03 and 9 iterations.



a) b)

Fig. 2. Mesh resulting from the solution of a) Fig. 1 and b) HR with equivalent detail threshold and the same number of iterations.

TABLE I
NUMBER OF DEGREES OF FREEDOM ADDED AFTER EACH ITERATION

	DT=0.05 (0.0016)		DT=0.03 (0.00058)		DT=0.015 (0.0001)	
IN	OCWB	HR	OCWB	HR	OCWB	HR
1	175	204	175	204	175	204
2	735	788	735	792	735	792
3	2892	2882	2985	3056	3007	3108
4	1263	1240	11091	9970	12137	11184
5	963	904	2403	2860	46155	44292
6	945	823	1620	2220	9072	NM
7	945	826	1545	2074	6390	NM
8	945	820	1539	2144	6009	NM

IN = Iteration Number. DT = Detail Threshold. NM = No Memory.

TABLE II
PROCESSING TIME (S)

	DT=0.05 (0.0016)		DT=0.03 (0.00058)		DT=0.015 (0.0001)	
NI	OCWB	HR	OCWB	HR	OCWB	HR
4	0.86	0.96	0.93	0.99	0.95	1.01
7	1.65	15.54	6.39	88.07	100.12	NM
9	1.97	32.49	7.06	207.86	103.57	NM

NI = Number of Iterations. The results were taken on a Core i7 2630QM PC with 16 GB RAM.

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